

# Contraction groups for tidy automorphisms of totally disconnected groups

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## Abstract

In this note, we show that results of U. Baumgartner and G. A. Willis concerning contraction groups of automorphisms of metrizable totally disconnected, locally compact groups remain valid also in the non-metrizable case, if one restricts attention to automorphisms for which small tidy subgroups exist.

*Key words:* locally compact group, totally disconnected group, scale function, scale, tidy subgroup, automorphism, contraction group, opposite subgroup, metrizability

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Given an automorphism  $\alpha$  of a locally compact group  $G$ , its contraction group  $U_\alpha$  is the group of all  $x \in G$  such that  $\lim_{n \rightarrow \infty} \alpha^n(x) = 1$ . Contraction groups arise in the study of semistable convolution semigroups on second countable groups [6], [8]. Contraction groups in  $p$ -adic Lie groups have been investigated in [2], and recently general results for metrizable totally disconnected groups  $G$  were obtained, using the concept of a tidy subgroup (see [10], [11]) as a tool [1]. In particular, it was shown in [1] that  $U_\alpha$  is closed if and only if  $G$  has small subgroups tidy for  $\alpha$ , and various reformulations and interesting consequences of this property were obtained. Let us say that  $\alpha$  is *tidy* if  $G$  has small subgroups tidy for  $\alpha$ . While every automorphism of a  $p$ -adic Lie group is tidy [1, Rem. 3.33 (2)], already for automorphisms of Lie groups over local fields of positive characteristic tidiness is an important additional regularity property, which is not always satisfied [4]. In this setting, the tidiness property characterizes those analytic automorphisms whose scale can be calculated on the Lie algebra level [4]. Presuming tidiness of the automorphisms involved, it is frequently possible to extend results for  $p$ -adic Lie groups (as in [3], [5] or [9]) to the case of positive characteristic [4].

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In this note, we describe consequences of the tidiness property. Based on techniques from [1], we establish two lemmas which allow us to transfer many of the results from [1] to non-metrizable groups, in the special case of tidy automorphisms. We also show that an automorphism  $\alpha$  is tidy if and only if its restriction  $\alpha|_{M_\alpha}$  to the “Levi factor”  $M_\alpha := \{x \in G : \alpha^{\mathbb{Z}}(x) \text{ is relatively compact}\}$  is tidy.

Throughout the following,  $G$  is a totally disconnected, locally compact topological group and  $\alpha \in \text{Aut}(G)$ . By [1, Prop.3.4],  $U_\alpha$  is a normal subgroup of  $P_\alpha := \{x \in G : \alpha^{\mathbb{N}}(x) \text{ is relatively compact}\}$ . Following [1], we set  $U_0 := \overline{U_\alpha} \cap \overline{U_{\alpha^{-1}}}$ . Given a subset  $H \subseteq G$  and  $x \in G$ , we write  $\lim_{n \rightarrow \infty} \alpha^n(x) = 1 \bmod H$  if, for every identity neighbourhood  $V \subseteq G$ , there exists  $N \in \mathbb{N}$  such that  $\alpha^n(x) \in VH$  for all  $n \geq N$ .<sup>2</sup> We write  $U_{\alpha/H} := \{x \in G : \lim_{n \rightarrow \infty} \alpha^n(x) = 1 \bmod H\}$ .

We recall that a compact, open subgroup  $V \subseteq G$  is called *tidy* for  $\alpha$  if

**T1**  $V = V_+ V_-$ , where  $V_\pm := \bigcap_{n \in \mathbb{N}_0} \alpha^n(V)$ ; and

**T2** The subgroups  $V_{++} := \bigcup_{n \in \mathbb{N}_0} \alpha^n(V_+)$  and  $V_{--} := \bigcup_{n \in \mathbb{N}_0} \alpha^{-n}(V_-)$  are closed in  $G$ .

Note that  $\alpha(V_+) \supseteq V_+$  and  $\alpha(V_-) \subseteq V_-$ . Thus  $V_- \subseteq P_\alpha$ . Tidy subgroups always exists, and the index  $s_G(\alpha) := [\alpha(V_+) : V_+]$  is independent of the choice of tidy subgroup  $V$  (see [10]). We abbreviate  $V_0 := V_+ \cap V_-$ . If  $V$  is tidy, then  $P_\alpha \cap V = V_-$  by [1, Rem.3.1], whence  $P_\alpha$  is closed. As a consequence, also  $M_\alpha = P_\alpha \cap P_{\alpha^{-1}}$  is closed, and  $V_0 = V \cap M_\alpha$  if  $V$  is tidy [10, La.9]. If  $H \leq G$  is an  $\alpha$ -stable closed subgroup (i.e.,  $\alpha(H) = H$ ) and  $V \subseteq G$  is tidy for  $\alpha$ , then there exists a subgroup  $W \subseteq G$  tidy for  $\alpha$  such that  $W \subseteq V$  and  $W \cap H$  is tidy for  $\alpha|_H$  [11, La.4.1]. Hence  $\alpha|_H$  is tidy if so is  $\alpha$ . We now adapt [1, Thm.3.8] to our setting (cf. [7] and [2, Thm.3.1], where analogous conclusions are achieved for real, resp.,  $p$ -adic Lie groups and compact  $H$ ).

**Lemma 1** *Let  $H$  be an  $\alpha$ -stable closed subgroup of  $G$ . If  $\alpha|_{M_\alpha}$  is tidy, then*

$$U_{\alpha/H} = U_\alpha H. \quad (1)$$

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<sup>2</sup>The definition in [1] only requires  $\alpha^n(x) \in VHV$ , but it is the above property which is really used in the proof of *loc. cit.* Theorem 3.8 (see p.9, line 2).

**Proof.** The inclusion  $U_\alpha H \subseteq U_{\alpha/H}$  is trivial. For the converse inclusion, let  $x \in U_{\alpha/H}$ ; we have to find  $h \in H$  such that  $xh \in U_\alpha$ . Let  $O$  be a compact, open subgroup of  $G$  tidy for  $\alpha$ . By [1, La.3.10], there exists  $h_0 \in H$  such that  $xh_0 \in U_{\alpha/H \cap O_0}$ . Since  $O_0 = M_\alpha \cap O$ , after replacing  $H$  with  $H \cap O_0$  and  $x$  with  $xh_0$  we may assume now without loss of generality that  $H \subseteq M_\alpha$  and  $H$  is compact. Let  $\mathcal{B}$  be the set of compact open subgroups of  $G$ . For each  $V \in \mathcal{B}$ , [1, La.3.10] yields  $h_V \in H$  and  $N_V \in \mathbb{N}$  such that  $\alpha^n(xh_V) \in V$  for all  $n \geq N_V$ . Since  $H$  is compact, the net  $(h_V)_{V \in \mathcal{B}}$  has a cluster point  $h \in H$ . We claim that  $\alpha^n(xh) \rightarrow 1$  as  $n \rightarrow \infty$ , i.e.,  $xh \in U_\alpha$ . Given an identity neighbourhood  $Y \subseteq G$ , there exists an identity neighbourhood  $W \subseteq G$  such that  $WW \subseteq Y$ ; since  $M_\alpha$  has small subgroups tidy for  $\alpha|_{M_\alpha}$ , after shrinking  $W$  we may assume that  $W \cap M_\alpha$  is a compact open subgroup of  $M_\alpha$  tidy for  $\alpha|_{M_\alpha}$ . Then  $H \cap W = H \cap M_\alpha \cap W = H \cap W_0$  is  $\alpha$ -stable. By definition of  $h$  as a cluster point, there exists  $V \in \mathcal{B}$  with  $V \subseteq W$  such that  $h_V^{-1}h \in W$  and thus  $h_V^{-1}h \in H \cap W$ . For all  $n \geq N_V$ , we now obtain

$$\alpha^n(xh) = \alpha^n(xh_V)\alpha^n(h_V^{-1}h) \in V\alpha^n(H \cap W) = V(H \cap W) \subseteq WW \subseteq Y.$$

Hence indeed  $\alpha^n(xh) \rightarrow 1$ . □

As a consequence of Lemma 1 just established, Proposition 3.7–Corollary 3.30 from [1] remain valid also for non-metrizable  $G$ , if one assumes in addition that  $\alpha|_{M_\alpha}$  be tidy. In fact, inspection shows that metrizability is never used directly in the proofs of these results, but only the validity of Eqn. (1) from above. The proof of Lemma 3.31 in *loc. cit.*, however, requires metrizability of  $G$  for a second reason. The lemma exploits an algorithm for the construction of tidy subgroups described in [12, §2]. This “tidying procedure” involves a certain subgroup,  $K$ , whose definition we presently recall. If  $G$  is metrizable, then  $K = \overline{U_\alpha \cap P_{\alpha^{-1}}}$  (see [12, p.4]), and this equality is used essentially in [1]. Unfortunately, it is not known whether equality persists for non-metrizable  $G$ . Therefore, to transfer [1, La.3.31 and Thm.3.32] to our setting, we first need to discuss  $K$  by hand, assuming that  $\alpha|_{M_\alpha}$  is tidy. Let us recall the definition from [12, p.4].

**Definition.** Write  $\mathcal{K}_O := \{x \in M_\alpha : \alpha^n(x) \in O \text{ for all sufficiently large } n\}$  and  $K_O := \overline{\mathcal{K}_O}$ , for each compact open subgroup  $O \subseteq G$ . We define  $K := \bigcap \{K_O : O \text{ a compact open subgroup of } G\}$ .

Note that  $\mathcal{K}_O = O_{--} \cap M_\alpha = (O \cap M_\alpha)_{--}$  here.

**Lemma 2** *If  $\alpha|_{M_\alpha}$  is tidy, then  $K = U_0 = \{1\}$ .*

**Proof.** Let us show first that  $K \subseteq \bigcap \{W_{--} : W \in \mathcal{T}(\alpha|_{M_\alpha})\}$ , where  $\mathcal{T}(\alpha|_{M_\alpha})$  is the set of subgroups  $W \subseteq M_\alpha$  tidy for  $\alpha|_{M_\alpha}$ . To this end, let  $W \in \mathcal{T}(\alpha|_{M_\alpha})$ . Then there exists a compact open subgroup  $O \subseteq G$  such that  $O \cap M_\alpha \subseteq W$ . Hence  $K_O = (O \cap M_\alpha)_{--} \subseteq W_{--}$  and thus  $K \subseteq K_O \subseteq W_{--}$ , using that  $W_{--}$  is closed. The assertion follows. Since  $M_\alpha = M_{\alpha|_{M_\alpha}}$ , we have  $W_- = W = W_+$  for each  $W \in \mathcal{T}(\alpha|_{M_\alpha})$  by [1, La. 3.19], whence  $\alpha(W) = W$  and thus  $W_{--} = W$ . Summing up, we have  $K \subseteq \bigcap \mathcal{T}(\alpha|_{M_\alpha})$ . But  $\bigcap \mathcal{T}(\alpha|_{M_\alpha}) = \{1\}$ , as  $\alpha|_{M_\alpha}$  is tidy. Thus  $K = \{1\}$ . Using [1, Cor. 3.27] and  $\overline{U_\alpha} \cap \overline{U_{\alpha^{-1}}} \subseteq P_\alpha \cap P_{\alpha^{-1}} = M_\alpha$ , we also get  $\{1\} = \bigcap \mathcal{T}(\alpha|_{M_\alpha}) = \overline{U_{\alpha|_{M_\alpha}}} \cap \overline{U_{\alpha^{-1}|_{M_\alpha}}} = \overline{U_\alpha} \cap \overline{U_{\alpha^{-1}}} = U_0$ .  $\square$

Stimulated by the results in [1] (notably Prop. 3.7, Prop. 3.21 and Thm. 3.32) and [9, Thm. 3.5 (iii)], we now formulate various useful conclusions:

**Theorem.** *Let  $G$  be a totally disconnected, locally compact group. Then  $\alpha \in \text{Aut}(G)$  is tidy if and only if  $\alpha|_{M_\alpha}$  is tidy. In this case, we have:*

- (a)  $U_0 = \{1\}$ .
- (b)  $U_\alpha$  and  $U_{\alpha^{-1}}$  are closed.
- (c)  $U_\alpha \cap M_\alpha = U_{\alpha^{-1}} \cap M_\alpha = \{1\}$ .
- (d) Every compact open subgroup of  $G$  satisfying **T1** is tidy.
- (e)  $P_\alpha = U_\alpha \rtimes M_\alpha$  and  $P_{\alpha^{-1}} = U_{\alpha^{-1}} \rtimes M_{\alpha^{-1}}$  as topological groups.
- (f)  $U_\alpha M_\alpha U_{\alpha^{-1}}$  is an  $\alpha$ -stable, open subset of  $G$  which contains every subgroup tidy for  $\alpha$ . The product map  $\mu: U_\alpha \times M_\alpha \times U_{\alpha^{-1}} \rightarrow U_\alpha M_\alpha U_{\alpha^{-1}}$ ,  $(x, y, z) \mapsto xyz$  is a homeomorphism.
- (g)  $s_G(\alpha^{-1}) = s_{P_\alpha}(\alpha^{-1}|_{P_\alpha}) = s_{U_\alpha}(\alpha^{-1}|_{U_\alpha}) = s_{V_{--}}(\alpha^{-1}|_{V_{--}})$  for each subgroup  $V \subseteq G$  tidy for  $\alpha$ , where each of  $P_\alpha$ ,  $V_{--}$  and  $U_\alpha$  are  $\alpha$ -invariant closed subgroups of  $P_\alpha$ .
- (h)  $s_H(\alpha^{-1}|_H) = \Delta_H(\alpha^{-1}|_H)$  for each  $\alpha$ -invariant closed subgroup  $H$  of  $P_\alpha$ , where  $\Delta_H: \text{Aut}(H) \rightarrow \mathbb{Q}^+$  is the modular function. In particular,  $s_G(\alpha^{-1}) = \Delta_{U_\alpha}(\alpha^{-1}|_{U_\alpha})$ .
- (i) If  $q: G \rightarrow Q$  is a quotient morphism with  $\alpha(\ker(q)) = \ker(q)$  and  $\bar{\alpha}$  the induced automorphism of  $Q$ , then  $q(U_\alpha) = U_{\bar{\alpha}}$ .

**Proof.** We already know that tidiness of  $\alpha$  entails tidiness of  $\alpha|_{M_\alpha}$ . We assume that  $\alpha|_{M_\alpha}$  is tidy now and show that that this entails all of (a)–(i) as well as tidiness of  $\alpha$ . We have already seen that (a) holds; (g), (h) and (i) are the analogues [1, Prop.3.7 and Prop.3.21] in our setting, which are remain valid by Lemma 1 above.

(b) Using (a) and [1, Cor.3.30], we obtain  $\overline{U_\alpha} = U_0 U_\alpha = \{1\} U_\alpha = U_\alpha$ . Replacing  $\alpha$  with  $\alpha^{-1}$  (whose restriction to  $M_\alpha = M_{\alpha^{-1}}$  is tidy since so is  $\alpha$ ), we see that also  $U_{\alpha^{-1}}$  is closed.

(c) By (a), (b) and [1, La.3.29], we have  $U_\alpha \cap M_\alpha = \overline{U_\alpha} \cap M_\alpha = U_0 = \{1\}$ . Replacing  $\alpha$  with  $\alpha^{-1}$ , we also obtain the second formula.

(d) Let  $V \subseteq G$  be a compact open subgroup satisfying **T1**. We apply the tidying procedure from [12, §2] to create a subgroup tidy for  $\alpha$ . Since  $V$  satisfies **T1**, Step 1 of the algorithm outputs  $V$ . In Step 2a, one defines  $K$  as above; thus  $K = \{1\}$ . In Step 3a, one defines  $V'' := \{v \in V : vxv^{-1} \in VK\}$ ; then  $V'' = V$  as  $K = \{1\}$ . Now  $V''K = V$  is tidy by [12, Prop.3.1].

$\alpha$  is tidy: Since every compact open subgroup of  $G$  contains a compact open subgroup satisfying **T1** by the first step of the algorithm just cited, and any such is tidy by (d), we see that  $G$  has small subgroups tidy for  $\alpha$ .

(e) Can be proved as [1, Thm.3.32 (6) and (6')].

(f) To see that  $\mu$  is injective, suppose that  $\mu(x, y, z) = \mu(x', y', z')$ . Then  $y^{-1}x^{-1}x'y' = z(z')^{-1} \in M_\alpha U_\alpha \cap U_{\alpha^{-1}} = P_\alpha \cap U_{\alpha^{-1}} = U_0 = \{1\}$  by [1, La.3.29], whence  $z = z'$  and  $xy = x'y'$ . Thus  $x^{-1}x' = y(y')^{-1} \in U_\alpha \cap M_\alpha = \{1\}$ , using (c), and hence  $x = x'$  and  $y = y'$ . Thus  $\mu$  is injective. Let  $V \subseteq G$  be tidy for  $\alpha$ . Then  $V = V_- V_+$  where  $V_- = (U_\alpha \cap V)V_0$  as a consequence of [1, Prop.3.16], and likewise  $V_+ = V_0(U_{\alpha^{-1}} \cap V)$ . Here  $U_\alpha \cap V$ ,  $U_{\alpha^{-1}} \cap V$  and  $V_0 = M_\alpha \cap V$  are compact open subgroups of  $U_\alpha$ ,  $U_{\alpha^{-1}}$  and  $M_\alpha$ , respectively. Since  $V = (U_\alpha \cap V_-)V_0(U_{\alpha^{-1}} \cap V)$ , the map  $\mu$  induces a continuous bijection between  $W := (U_\alpha \cap V) \times (M_\alpha \cap V) \times (U_{\alpha^{-1}} \cap V)$  and  $V$ . Both  $W$  and  $V$  being compact, we deduce that  $\mu|_W^V$  is a homeomorphism. To verify that  $\text{im}(\mu)$  is open and  $\mu$  a homeomorphism, let  $(x_0, y_0, z_0) \in U_\alpha \times M_\alpha \times U_{\alpha^{-1}}$ . As  $M_\alpha$  normalizes  $U_\alpha$ , the formula  $h(x, y, z) := (y_0^{-1}xy_0, y, z)$  defines a homeomorphism  $h$  of  $U_\alpha \times M_\alpha \times U_{\alpha^{-1}}$  onto itself. Left and right translations being homeomorphisms and  $h^{-1}(W)$  being an identity neighbourhood, we now deduce from

$$x_0 y_0 \mu|_W(h(x, y, z)) z_0 = \mu(x_0 x, y_0 y, z z_0) \quad \text{for all } (x, y, z) \in h^{-1}(W)$$

that  $\mu$  takes a neighbourhood of  $(x_0, y_0, z_0)$  homeomorphically onto a neighbourhood of  $\mu(x_0, y_0, z_0)$ .  $\square$

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